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Discrete Mathematics 284 (2004) 11–20
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Tight incomplete block designs

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Received 25 March 2003; received in revised form 28 April 2003; accepted 21 November 2003

Dedicated to Charles C. Lindner on the occasion of his 65th birthday

Abstract

An *incomplete t-wise balanced design* (ItBD) of type t-$(v,h,\mathscr{K},\lambda)$ is a triple $(X,H,\mathscr{B})$ where X is a v-element set of *points*, H is an h-element subset $H \subseteq X$ called the *hole*, and $\mathscr{B}$ is a collection of subsets of X called *blocks*, such that the size of every block $B \in \mathscr{B}$ is in $\mathscr{K}$ and every t-element subset of X is either in the hole or in exactly λ blocks, but not both. Kreher and Rees (Codes an Designs, Ohio state University Research Institute Publication, 10 (2002) 179) derived an upper bound on the size of the hole, which is given here in Theorem 3. An ItBD meeting this bound is called a *tight incomplete block design*. In this paper we study the existence of tight incomplete block designs whose automorphism group is as large as possible. In particular, we obtain a characterization of those tight ItBDs $(X,H,\mathscr{B})$ of prime-power index λ admitting $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group.

Keywords: t-Designs; Designs with holes

1. Introduction

A *t-wise balanced design* (tBD) of type t-$(v,\mathscr{K},\lambda)$ is a pair $(X,\mathscr{B})$ where X is a v-element set of *points* and $\mathscr{B}$ is a collection of subsets of X called *blocks*, with the property that the size of every block $B \in \mathscr{B}$ is in $\mathscr{K}$ and every t-element subset of X is contained in exactly λ blocks. If $\mathscr{K}$ is a set of positive integers strictly between t and v, then we say that the tBD is *proper*. If $\lambda = 1$, the notation $S(t,\mathscr{K},v)$ is often used and the design is called a *Steiner* tBD.

An *incomplete t-wise balanced design* (ItBD) of type t-$(v,h,\mathscr{K},\lambda)$ is a triple $(X,H,\mathscr{B})$ where X is a v-element set of *points*, H is an h-element subset $H \subseteq X$ called the *hole*, and $\mathscr{B}$ is a collection of subsets of X called *blocks*, such that the size of every block $B \in \mathscr{B}$ is in $\mathscr{K}$ and every t-element subset of X is either in the hole or in exactly λ blocks, but not both. In 2001, Kreher and Rees [8] proved the following result for incomplete t-wise balanced designs:

Theorem 1 (Kreher and Rees [8,9]). *If H is a hole in an* ItBD *with $t \geqslant 2$ and any λ, then*

$$|H| \leqslant \frac{v-1}{2} \quad \textit{for } t \textit{ even,}$$

while

$$|H| \leqslant \frac{v}{2} \quad \textit{for } t \textit{ odd.}$$

Notice that a Steiner tBD is an incomplete t-wise balanced design in which any block can be considered as the hole. Thus if we set $\lambda = 1$ in Theorem 1, then we get following corollary. This establishes the validity of Kramer's conjecture [7] for all $t \geqslant 2$.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2003.11.020

Corollary 2 (Kreher and Rees [8,9]). *If B is a block in a Steiner tBD, then*

$$|B| \leqslant \frac{v-1}{2} \quad \textit{for } t \textit{ even},$$

while

$$|B| \leqslant \frac{v}{2} \quad \textit{for } t \textit{ odd}.$$

Kreher and Rees showed that these bounds for ItBD's given in Theorem 1 are sharp infinitely often. Unfortunately, the construction has large index λ. They also showed that if these bounds are sharp, then blocks of size $t+1$ are required. This motivated the study of ItBD's in which the minimum size of a block is specified. In [9] Kreher and Rees derived an upper bound on the size of a hole in an incomplete t-wise balanced design with specified minimum size of a block.

Theorem 3 (Kreher and Rees [8,9]). *If $(X, H, \mathscr{B})$ is a proper* ItBD *of type t-$(v, h, \mathscr{K}, \lambda)$ with $h \geqslant t \geqslant 2$ and $\min \mathscr{K} = k \geqslant t+1$, then*

$$h \leqslant \frac{v+(k-t)(t-2)-1}{k-t+1}.$$

It was also shown in [9] that this bound is sharp for $t=2$ or 3. More precisely, for each $h \geqslant t$ and each $k \geqslant t+1$, $(t, h, k) \neq (3, 3, 4)$, there exists an ItBD meeting the bound for some $\lambda = \lambda(t, h, k)$. In [1] it was shown that this bound is asymptotically sharp for all t.

We say that an ItBD meeting the bound in Theorem 3 is a *tight* incomplete block design. The condition of being tight carries over to the derived design. If S is a subset of points contained in the hole of an ItBD$(X, H, \mathscr{B})$, then the *derived design* with respect to S has as blocks

$$\{B \setminus S : S \subseteq B \in \mathscr{B}\}.$$

We have the following theorem.

Lemma 4. *If $(X, H, \mathscr{B})$ is a tight* ItBD *of type t-$(v, h, \mathscr{K}, \lambda)$ and $S \subseteq H$ with $|S| = s$ where $1 \leqslant s \leqslant t-2$, then the derived design with respect to S is a tight* I$(t-s)$BD *of type $(t-s)$-$(v-s, h-s, \mathscr{K}-s, \lambda)$.*

Proof. We have

$$h = \frac{v+(k-t)(t-2)-1}{k-t+1}$$

which we rewrite as

$$v-h-1 = (h-t+2)(k-t). \tag{1}$$

Then

$$\begin{aligned}(v-s)-(h-s)-1 &= v-h-1 \\ &= (h-t+2)(k-t) \\ &= ((h-s)-(t-s)+2)((k-s)-(t-s)).\end{aligned}$$

i.e.

$$h-s = \frac{(v-s)+((k-s)-(t-s))((t-s)-2)-1}{(k-s)-(t-s)+1},$$

hence the derived design is also tight. □

If G is the automorphism group of an ItBD$(X, H, \mathscr{B})$, then it must leave invariant the hole H and the points $X \setminus H$ that are outside the hole. Thus G must be a subgroup of $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$. In the next section we examine the possibility of obtaining a tight ItBD of type t-(v, h, k, λ) whose automorphism group G is as large as possible, i.e. when $G = \mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$.

2. Tight designs from $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$

Let X be a v-element set of points and $H \subset X$ have $|H| = h$. If B is a block of an ItBD $(X, H, \mathscr{B})$ of type t-(v, h, k, λ), then $|B \cap H| = j$ for some $j = 0, 1, \ldots, t-1$. Thus we say that a n-element subset that intersects the hole H in j points is a subset of type $(j, n-j)$ and we set

$$\Gamma_j = \{K \subseteq X : K \text{ is a type } (j, k-j) \textit{subset}\}$$

and

$$\Delta_i = \{T \subseteq X : T \text{ is a type } (i, t-i) \textit{subset}\},$$

where $0 \leqslant i, j \leqslant t-1$. The number of sets in Γ_j that contain a given member $T \in \Delta_i$ is

$$M_t[i,j] = |\{B \in \Gamma_j : T \subseteq B\}| = \binom{h-i}{j-i}\binom{v-h-t+i}{k-t-j+i}. \tag{2}$$

Notice that M_t is a square nonsingular upper triangular matrix, thus the matrix equation

$$M_t \vec{u} = J_t \tag{3}$$

has a unique solution vector $\vec{u} = [u_0, u_1, \ldots, u_{t-1}]^{\mathrm{T}}$. The following result is easily established.

Lemma 5. *Let M_t be the matrix defined in Eq. (2), where t, v, h and k are the parameters of a tight block design. Then the last two entries of the solution vector $\vec{u}$ to Eq. (3) are $u_{t-1} = \binom{v-h-1}{k-t}^{-1}$ and $u_{t-2} = 0$.*

Proof. Solving Eq. (3) we get

$$u_{t-1} = \binom{v-h-1}{k-t}^{-1},$$

$$\begin{aligned} u_{t-2} &= \binom{v-h-2}{k-t}^{-1}\left\{1 - (h-t+2)\binom{v-h-2}{k-t-1}u_{t-1}\right\} \\ &= \binom{v-h-2}{k-t}^{-1}\left\{1 - \frac{(h-t+2)(k-t)}{v-h-1}\right\} \\ &= 0, \end{aligned}$$

because in a tight design $v-h-1 = (h-t+2)(k-t)$ by Eq. (1). $\square$

If we consider a t-(v, h, k, λ) design $(X, H, \mathscr{B})$ with $G = \mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group, then $\mathscr{B}$ is a union of G-orbits of k-element subsets. The Γ_js defined above are the orbits of k-element subsets. If w_j is the number of times each block of orbit Γ_j appears in the design then

$$M_t \vec{w} = \lambda J_t,$$

where $\vec{w} = [w_0, w_1, \ldots, w_{t-1}]$. Hence $\vec{w} = \lambda \vec{u}$. Thus we have Lemmas 6 and 7.

Lemma 6. *If there exists a t-(v, h, k, λ) design with $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group, then $\lambda \vec{u}$ must be nonnegative and integral, where $\vec{u}$ is the unique solution to $M_t \vec{u} = J_t$.*

Lemma 7. *If a t-(v, h, k, λ) design exists with $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group, then it has $|\Gamma_j| \lambda u_j$ blocks of type $(j, k-j)$, where $|\Gamma_j| = \binom{h}{j}\binom{v-h}{k-j}$ and $M_t \vec{u} = J_t$.*

The next observation is a simple consequence of Lemmas 5 and 7.

Corollary 8. *In a tight t-(v,h,k,λ) design with* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group there are no blocks of type* $(t-2,k-t+2)$.

Proof. Lemma 5 shows that $u_{t-2}=0$, thus applying Lemma 7 we see that there are $0=|\Gamma_{t-2}|\lambda u_{t-2}$ blocks of type $(t-2,k-t+2)$. □

Theorem 9. *There are no proper tight t-$(v,h,k,1)$ designs with* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group.*

Proof. An ItBD with $\lambda=1$ having $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ as an automorphism group exists if and only if $M_t\vec{u}=J_t$ with $\vec{u}$ nonnegative and integral, where the orbit incidence matrix M_t is given in Eq. (2). In particular, let us consider the last coordinate $u_{t-1}=\binom{v-h-1}{k-t}^{-1}$. This is an integer if and only if $\binom{v-h-1}{k-t}^{-1}=1$. Therefore, either $k-t=v-h-1$ or $k-t=0$. If $k-t=0$, then $k=t$ and the design is not proper.

Suppose $k-t=v-h-1$ and recall that the design is tight whenever

$$h=\frac{v+(k-t)(t-2)-1}{k-t+1}.$$

Therefore,

$$\begin{aligned}h&=\frac{v+(v-h-1)(t-2)-1}{v-h-1+1}\\&=\frac{v+tv-ht-t-2v+2h+2-1}{v-h}\\&=\frac{v(t-1)-h(t-2)-t+1}{v-h}\end{aligned}$$

and we have the following sequence of equations:

$$v(t-1)-h(t-2)-t+1=vh-h^2,$$

$$h^2-vh+v(t-1)-h(t-2)=t-1,$$

$$(h-v)(h-(t-1))+h=t-1,$$

$$(h-v)(h-(t-1))+(h-(t-1))=0,$$

$$(h-(t-1))(h-v+1)=0.$$

Therefore,

$$h=t-1 \quad\text{or}\quad h=v-1.$$

(i) Suppose $h=t-1$. Then

$$k-t=v-h-1=v-(t-1)-1=v-t.$$

Hence, $k=v$ and the design is not proper.

(ii) Suppose $h=v-1$. Then

$$k-t=v-h-1=v-(v-1)-1=0.$$

Hence, $k=t$ and again the design is not proper.

Therefore there are no proper Steiner ItBDs with this automorphism group. □

The following result appeared in the 1968 paper by Hering [5]. However, there are some difficulties with its proof. We provide below a correct one.

Theorem 10. *If $\binom{n}{k}=p^\alpha$, where p^α is a prime power, then $n=p^\alpha$ and either $k=1$ or $k=p^\alpha-1$.*

Proof. Suppose $n,k \geqslant 0$ are integers such that $\binom{n}{k} = p^\alpha$ for some prime p and positive integer α. Without loss of generality we can suppose that $k \leqslant n-k$. Thus $n \geqslant 2k$. Denote by a_i the number of integers in $\{n-k+1, n-k+2, \ldots, n\}$ that are multiples of p^i, and denote by b_i the number of integers in $\{1,2,\ldots,k\}$ that are multiples of p^i. Then

$$a_i = |\{(n-t): p^i|(n-t), t = 0,1,\ldots,k-1\}|$$

and

$$b_i = |\{(k-t): p^i|(k-t), t = 0,1,\ldots,k-1\}|.$$

Let u, v be the unique integers such that

$$p^u \leqslant n < p^{u+1} \quad \text{and} \quad p^v \leqslant k < p^{v+1}. \tag{4}$$

First consider the case when $p > k$. Then $b_i = 0$, $a_i \leqslant 1$ for all $i \geqslant 1$, and $a_i = 0$ for $i > u$. Therefore

$$\alpha = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} a_i \leqslant \sum_{i=1}^{u} 1 = u$$

and so

$$\frac{\binom{n}{k}}{p^\alpha} \geqslant \frac{\binom{n}{k}}{p^u} = \frac{n/k\binom{n-1}{k-1}}{p^u}.$$

Since $n \geqslant p^u$ and $n \geqslant 2k$, we have

$$\frac{\binom{n}{k}}{p^\alpha} \geqslant \frac{n}{p^u}\frac{\binom{n-1}{k-1}}{k} \geqslant \frac{\binom{n-1}{k-1}}{k} \geqslant \frac{\binom{2k-1}{k-1}}{k}.$$

For $2 \leqslant k \leqslant n/2$ we have

$$\frac{\binom{2k-1}{k-1}}{k} \geqslant \frac{\binom{2k-1}{1}}{k} = \frac{2k-1}{k} = 2 - \frac{1}{k} > 1.$$

Therefore $\binom{n}{k}/p^\alpha > 1$ and hence $\binom{n}{k} \neq p^\alpha$, a contradiction. Thus the only remaining possibility is for $k = 1$. In this case $n = p^\alpha$ and the result follows.

To attack the case when $p \leqslant k$ we use the fact that

$$a_i = \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor \quad \text{and} \quad b_i = \left\lfloor \frac{k}{p^i} \right\rfloor$$

to obtain the following bounds on a_i and b_i:

$$a_i < \frac{k}{p_i} + 1 \quad \text{for } i \leqslant u, \quad a_i = 0 \quad \text{for } i > u,$$

$$b_i > \frac{k}{p_i} - 1 \quad \text{for} \quad i \leqslant v, \quad b_i = 0 \quad \text{for } i > v.$$

Now we bound α in the following way:

$$\alpha < \sum_{i=1}^{u}\left(\frac{k}{p_i}+1\right) - \sum_{i=1}^{v}\left(\frac{k}{p_i}-1\right) = \frac{k}{p^{v+1}}\sum_{i=0}^{u-v-1}\frac{1}{p^i} + u + v.$$

From Eq. (4) we have that $k/p^{v+1} < 1$. Also, from well-known properties of the infinite geometric series, namely

$$\sum_{i=0}^{k} x^i < \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad \text{for } 0 < x < 1,$$

we have that $\sum_{i=0}^{u-v-1} 1/p^i < 1 + 1/p - 1$. As a result, we obtain

$$\alpha < u + v + 1 + \frac{1}{p-1}.$$

From Eq. (4) we have

$$\frac{n}{p^u} \geqslant 1 \quad \text{and} \quad \frac{1}{p^v} \geqslant \frac{1}{k}.$$

Elementary calculus shows that $p^{1/(p-1)} \leqslant 2$ for all $p \geqslant 2$. Thus as $p \leqslant k$ we have

$$\frac{\binom{n}{k}}{p^{\alpha}} \geqslant \frac{n/k\binom{n-1}{k-1}}{p^{u}\cdot p^{v}\cdot p\cdot p^{1/(p-1)}} \geqslant \frac{\binom{n-1}{k-1}}{2pk^{2}} \geqslant \frac{\binom{n-1}{k-1}}{2k^{3}}. \tag{5}$$

For $k \geqslant 6$ and $n \geqslant 2k \geqslant k+6$ we have

$$\frac{\binom{n}{k}}{p^{\alpha}} \geqslant \frac{\binom{n-1}{k-1}}{2k^{3}} \geqslant \frac{\binom{k+5}{5}}{2k^{3}} = \frac{(k+5)(k+4)(k+3)(k+2)(k+1)}{2\cdot 5!\cdot k^{3}} > 1,$$

by Eq. (5). This means that $\binom{n}{k} \neq p^{\alpha}$, again a contradiction.

For $k = 3, 4, 5$ and $n \geqslant 12$ we see from Eq. (5) that

$$\frac{\binom{n}{k}}{p^{\alpha}} \geqslant \frac{\binom{11}{4}}{2\cdot 5^{3}} = \frac{330}{250} > 1 \quad \text{for } k = 5,$$

$$\frac{\binom{n}{k}}{p^{\alpha}} \geqslant \frac{\binom{11}{3}}{2\cdot 4^{3}} = \frac{165}{128} > 1 \quad \text{for } k = 4, \text{ and}$$

$$\frac{\binom{n}{k}}{p^{\alpha}} \geqslant \frac{\binom{11}{2}}{2\cdot 3^{3}} = \frac{55}{54} > 1 \quad \text{for } k = 3.$$

Each leads to the contradiction that $\binom{n}{k} \neq p^{\alpha}$. The beginning assumption was that $n \geqslant 2k$. Therefore we have to also consider $n = 10$ and 11 for $k = 5$, $n = 8, 9, 10$ and 11 for $k = 4$ and $n = 6, 7, 8, 9, 10$ and 11 for $k = 3$. However, all the resulting binomials ($\binom{10}{5}$, $\binom{11}{5}$, $\binom{8}{4}$, $\binom{9}{4}$, $\binom{10}{4}$, $\binom{11}{4}$, $\binom{6}{3}$, $\binom{7}{3}$, $\binom{8}{3}$, $\binom{9}{3}$, $\binom{10}{3}$ and $\binom{11}{3}$) are not prime powers.

When $k = 2$ the result is obvious, because $\binom{n}{2} = n(n-1)/2$ can never be a prime power when $n \geqslant 4$.

Thus the only remaining possibility is for $k = 1$ and the result follows. □

It should also be noted that Erdös in [4] established a similar result. Namely:

Theorem 11 (Erdös [4]). *The binomial coefficient $\binom{n}{k}$ is never m^{ℓ} for any integer m, when $\ell \geqslant 2$ and $4 \leqslant k \leqslant n-4$.*

Theorem 12. *If $\lambda = p^{\alpha}$, where p is a prime, then there are no proper tight t-(v, h, k, λ) designs with $t \geqslant 4$ and $G = \mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group.*

Proof. An ItBD with G as an automorphism group exists if and only if $M_t\vec{u} = J_t$ with $\lambda\vec{u}$ nonnegative and integral (see Lemma 6), where the orbit incidence matrix M_t is given in Eq. (2). In particular, consider the last coordinate $u_{t-1} = \binom{v-h-1}{k-t}^{-1}$. Because $\lambda = p^{\alpha}$, a prime power, it must be either integral or equal to $1/p^{\beta}$, where β is a positive integer and $1 \leqslant \beta \leqslant \alpha$. We have shown in Theorem 9 that u_{t-1} cannot be an integer, so we need to consider only the case when $u_{t-1} = 1/p^{\beta}$. Therefore, $\binom{v-h-1}{k-t} = p^{\beta}$ which implies, by Lemma 10, that either $v-h-1 = p^{\beta}$ and $k-t = 1$, or $v-h-1 = p^{\beta}$ and $k-t = p^{\beta}-1$. Let us consider these two cases.

(i) Suppose $v-h-1 = p^{\beta}$ and $k-t = 1$, which implies that $v = h+p^{\beta}+1$ and $k = t+1$. Now recall that the design is tight whenever

$$h = \frac{v+(k-t)(t-2)-1}{k-t+1}.$$

Therefore,

$$h = \frac{(h+p^{\beta}+1)+(t-2)-1}{1+1}$$

and hence

$$h = t+p^{\beta}-2.$$

(ii) Suppose $v-h-1 = p^{\beta}$ and $k-t = p^{\beta}-1$, which implies that $v = h+p^{\beta}+1$ and $k = t+p^{\beta}-1$. Again recall that the design is tight whenever

$$h = \frac{v+(k-t)(t-2)-1}{k-t+1}.$$

Therefore,

$$h = t - \frac{p^\beta - 2}{p^\beta - 1}$$

which is impossible for all $p^\beta > 2$, because h and t must be integers. For $p^\beta = 2$ we have $h = t$, which is the same as in case (i).

Therefore, for any prime power p^α, such that $\lambda = p^\alpha$, if there exists a tight t-(v, h, k, λ) design with G as an automorphism group, then it must have the following parameters:

$$\begin{aligned} &k = t + 1, \\ &h = t + (p^\beta - 2), \\ &v = h + (p^\beta + 1), \end{aligned} \tag{6}$$

where $1 \leqslant \beta \leqslant \alpha$. However, any I$t$BD must also satisfy the condition of Theorem 1. Namely:

(a) For even t we must have $2h + 1 \leqslant v$. Using Eq. (6), we can get that

$$\begin{aligned} &2h + 1 \leqslant h + (p^\beta + 1), \\ &h \leqslant p^\beta, \\ &t + (p^\beta - 2) \leqslant p^\beta, \\ &t \leqslant 2. \end{aligned}$$

(b) For odd t we must have $2h \leqslant v$. Again, using Eq. (6), we can get that

$$\begin{aligned} &2h \leqslant h + (p^\beta + 1), \\ &h \leqslant p^\beta + 1, \\ &t + (p^\beta - 2) \leqslant p^\beta + 1, \\ &t \leqslant 3. \end{aligned}$$

Therefore, $t \leqslant 3$ and the result follows. □

Theorem 13. *There are no proper tight t-$(v, h, k, 2)$ designs with $t \geqslant 3$ and $G = \mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group.*

Proof. Since $\lambda = 2$ is a prime, if there exists a proper tight t-$(v, h, k, 2)$ design with G as an automorphism group, then by Theorem 12 $t = 2$ or 3. However, for $t = 3$, the coefficient u_{t-3} of the solution vector $\vec{u}$ of the matrix equation $M_t \vec{u} = J_t$, given in Eq. (3), must satisfy

$$\binom{v - h - 3}{k - t} u_{t-3} = \frac{[h - (t-1)][k - (t-1)]}{2(v - h - 2)}. \tag{7}$$

By Eq. (6), the parameters of such a design must have the form: $k = t + 1$, $h = t$ and $v = h + 3$, i.e. $t = 3$, $h = 3$, $k = 4$ and $v = 6$. Then

$$\binom{v - h - 3}{k - t} u_{t-3} = \binom{3 - 3}{1} u_{t-3} = \binom{0}{1} u_{t-3} = 0.$$

But on the other hand

$$\frac{[h - (t-1)][k - (t-1)]}{2(v - h - 2)} = \frac{[t - (t-1)][t + 1 - (t-1)]}{2(3 - 2)} = \frac{2}{2} = 1.$$

This contradicts Eq. (7).

Therefore, we cannot construct a proper tight 3-$(v,h,k,2)$ design with G as an automorphism group, and the result follows. □

Theorem 14. *If $\lambda = p^\alpha$, where $p \geqslant 5$ is a prime, then there are no proper tight t-(v,h,k,λ) designs with $t \geqslant 3$ and $G = \mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group.*

Proof. Suppose $t \geqslant 3$ and $\lambda = p^\alpha$, where $p \geqslant 5$ is a prime. If there exists a proper tight t-(v,h,k,λ) design with $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group, then by Theorem 12 $t = 2$ or 3. Suppose there exists a 3-(v,h,k,λ) design with G as an automorphism group. Then, by Lemma 6, there exists a nonnegative, integral solution $\vec{w}$ to the matrix equation $M_3\vec{w} = \lambda J_3$, where the $[i,j]$-entry of the matrix M_3 is given by

$$M_3[i,j] = |\{B \in \Gamma_j : T \subseteq B\}| = \binom{h-i}{j-i}\binom{v-h-3+i}{k-3-j+i}.$$

Since the parameters of this design must satisfy conditions from Eq. (6), we have $k = 4$, $h = p^\beta + 1$ and $v = 2(p^\beta + 1)$, where $1 \leqslant \beta \leqslant \alpha$. So the matrix M_3 has the form

$$M_3 = \begin{bmatrix} p^\beta - 2 & p^\beta + 1 & 0 \\ 0 & p^\beta - 1 & p^\beta \\ 0 & 0 & p^\beta \end{bmatrix}$$

and

$$\begin{bmatrix} p^\beta - 2 & p^\beta + 1 & 0 \\ 0 & p^\beta - 1 & p^\beta \\ 0 & 0 & p^\beta \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} p^\alpha \\ p^\alpha \\ p^\alpha \end{bmatrix},$$

because $\lambda = p^\alpha$. Solving this equation we obtain $w_2 = p^\alpha/p^\beta = p^{\alpha-\beta}$, $w_1 = 0$ and $w_0 = p^\alpha/p^\beta - 2$, which is an integer only for $p=2$, $\beta=2$, $\alpha \geqslant 2$ or $p=3$, $\beta=1$ and $\alpha \geqslant 1$. Therefore, for $p \geqslant 5$, w_0 can never be an integer, and consequently there does not exist a tight 3-(v,h,k,λ) design with G as an automorphism group and $\lambda = p^\alpha$, where $p \geqslant 5$ is a prime. □

After analyzing the results from Theorems 13 and 14 we can conclude that the only possible 3-(v,h,k,λ) design with $\lambda = p$ (prime) and $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ as an automorphism group is a 3-$(8,4,4,3)$ design. This design is constructed in the proof of Lemma 15.

Lemma 15. *There exist proper tight* I3BD*s of types* 3-$(8,4,4,3)$ *and* 3-$(10,5,4,4)$ *each with* $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ *as an automorphism group.*

Proof. To construct a tight 3-$(8,4,4,3)$ design let $X = \{a,b,c,d,1,2,3,4\}$ be the set of points and $H = \{a,b,c,d\}$ be the hole. Take as the set of blocks

$$\mathscr{B} = \{xyij : x, y \in \{a,b,c,d\}, i,j \in \{1,2,3,4\}, x \neq y, i \neq j\}$$
$$\cup \{1234, 1234, 1234\}.$$

For a tight 3-$(10,5,4,4)$ design let $X = \{a,b,c,d,e,1,2,3,4,5\}$ be the set of points and $H = \{a,b,c,d,e\}$ be the hole. Take as the set of blocks

$$\mathscr{B} = \{xyij : x, y \in \{a,b,c,d,e\}, i,j \in \{1,2,3,4,5\}, x \neq y, i \neq j\}$$
$$\cup \{1234, 1235, 1245, 1345, 2345, 1234, 1235, 1245, 1345, 2345\}. \quad \square$$

Lemma 16. *If there exists a proper tight* ItBD *of type* t-(v,h,k,λ) *with* $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ *as an automorphism group, then there exists a proper tight* ItBD *of type* t-$(v,h,k,\lambda m)$ *with* $\mathrm{Sym}(H) \times \mathrm{Sym}(X \setminus H)$ *as an automorphism group for every integer* $m \geqslant 1$.

Proof. Let A be a tight t-(v,h,k,λ) design. First notice that the condition for a design to be tight

$$h = \frac{v+(k-t)(t-2)-1}{k-t+1},$$

does not depend on λ. Thus, we obtain a tight t-$(v,h,k,\lambda m)$ design by taking m copies of each block of A. □

The next Corollary 17 is a simple consequence of Lemmas 15 and 16.

Corollary 17. *For any positive integer m there exist proper tight* I3BD*s of types* 3-$(8,4,4,3m)$ *and* 3-$(10,5,4,4m)$ *each with* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group.*

The next observation, contained in Lemma 18, is that there exists a tight 2-(v,h,k,λ) design, with $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ as an automorphism group, for any prime power λ.

Lemma 18. *For any prime p and any $\alpha\geqslant 1$ there exists a proper tight* I2BD *of type* 2-$(2p^\alpha+1, p^\alpha, 3, p^\alpha)$ *with* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group.*

Proof. Suppose $t=2$ and $\lambda=p^\alpha$, where p is a prime. Let us consider a design with $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ as an automorphism group. If there exists a proper tight 2-(v,h,k,p^α) design with such an automorphism group, then by Lemma 6 there exists a non-negative, integral solution $\vec{w}$ to the matrix equation $M_2\vec{w}=\lambda J_2$, where the $[i,j]$-entry of the matrix M_2 is given by

$$M_2[i,j] = |\{B\in\Gamma_j\colon\ T\subseteq B\}| = \binom{h-i}{j-i}\binom{v-h-2+i}{k-2-j+i}.$$

Since the parameters of this design must satisfy conditions from Eq. (6), we have $k=3$, $h=p^\beta$ and $v=2p^\beta+1$, where $1\leqslant\beta\leqslant\alpha$. So the matrix M_2 has the form

$$M_2 = \begin{bmatrix} p^\beta-1 & p^\beta \\ 0 & p^\beta \end{bmatrix}$$

and

$$\begin{bmatrix} p^\beta-1 & p^\beta \\ 0 & p^\beta \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} p^\alpha \\ p^\alpha \end{bmatrix},$$

because $\lambda=p^\alpha$. Solving this equation we obtain $w_1=p^{\alpha-\beta}$ and $w_0=0$. Now set $\beta=\alpha$. Therefore, if $X=\{1,2,\ldots,2p^\alpha+1\}$ is a set of points, $H=\{1,2,\ldots,p^\alpha\}$, $H\subset X$ is the hole, and the set of blocks $\mathscr{B}$ consists of all 3-subsets intersecting the hole in exactly one point, i.e.

$$\mathscr{B} = \{xij{:}x\in H, i,j\in X\setminus H, i\neq j\},$$

then we get a tight 2-$(2p^\alpha+1, p^\alpha, 3, p^\alpha)$ design. □

Corollary 19. *There exists a proper tight* I2BD *of type* 2-$(2p^\alpha+1, p^\alpha, 3, mp^\alpha)$ *with* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group, for every prime p and all positive integers α, m.*

We are now ready to give our main result.

Theorem 20. *There exists a tight incomplete block design of type t-(v,h,k,λ), with λ a prime power, having* $\mathrm{Sym}(H)\times\mathrm{Sym}(X\setminus H)$ *as an automorphism group if and only if either $t=3$ and* $(v,h,k,\lambda)\in\{(8,4,4,3^\alpha){:}\alpha\geqslant 1\}\cup\{(10,5,4,2^\alpha){:}\alpha\geqslant 2\}$ *or $t=2$ and* $(v,h,k,\lambda)\in\{(2p^\alpha+1, p^\alpha, 3, p^\gamma){:}\gamma\geqslant\alpha\geqslant 1\}$.

Proof. Combine Theorems 9, 12, 13, 14 and Corollaries 17, 19. □

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